

Signatures of many-body localisation in a system without disorder and the relation to a glass transition

James M. Hickey,¹ Sam Genway,¹ and Juan P. Garrahan¹

¹*School of Physics and Astronomy, University of Nottingham, Nottingham, NG7 2RD, UK*

(Dated: May 23, 2014)

We study a quantum spin system with local bilinear interactions and without quenched disorder which seems to display characteristic signatures of a many-body localisation (MBL) transition. From direct diagonalisation of small systems, we find a change in certain dynamical and spectral properties at a critical value of a coupling, from those characteristic of a thermalising phase to those characteristic of a MBL phase. The system we consider is known to have a quantum phase transition in its ground-state in the limit of large size, related to a first-order active-to-inactive phase transition in the stochastic trajectories of an associated classical model of glasses. Our results here suggest that this transition is present throughout the spectrum of the system in the large size limit. These findings may help understand the connection between MBL and structural glass transitions.

For over half a century, it has been understood that a single quantum particle can become localised in space in the presence of a disordered potential [1, 2]. Over the last few years, there has been a surge of interest in localisation in the context of interacting many-body systems [3–6]. When a closed interacting system exhibits many-body localisation (MBL), a breakdown of thermalisation [7] occurs: the system is unable to function as its own thermal bath, dynamics retains memory of its initial state, and expectation values of observables do not relax to the values expected from thermal equilibrium [6]. Since the MBL transition is seen to occur in eigenstates with energies far above the ground state energy, it has been dubbed a “high-temperature” quantum phase transition [4]. To date, MBL has been demonstrated mostly in systems with quenched disorder, and it is currently of interest [6] to establish whether MBL is exhibited in quantum systems where the Hamiltonian itself is translationally invariant. Previous works have indicated that such effects may be possible in systems with both fast- and slow-moving particles [8], where the slow particles provide an effective disordered potential in which the fast particles can appear localised. Whether localisation behaviour can be seen without such impurities providing effective disorder is an open question [6].

In this paper we present and study a quantum spin system which has local interactions and is free of disorder that appears to show the characteristic features of MBL. The Hamiltonian we consider is related to a deformation of the master operator of a classical glass model, which is known [9] to display a (first-order) phase transition in its largest eigenvalue: in the quantum context this corresponds to a quantum phase-transition in the ground-state, while in the classical stochastic context it corresponds to a singularity in the cumulant generating function of the dynamical activity [10, 11] and thus an indication of a non-equilibrium transition to an inactive or glass state [9, 12]. We show below that this phase transition seems to be present throughout the spectrum

of the Hamiltonian and possesses many of the hallmarks of a MBL transition.

The system we study consists of spins or qubits on the sites of a one-dimensional lattice with periodic boundary conditions. The Hamiltonian has the general form,

$$H = -\frac{1}{2} \sum_{i=1}^N (\Gamma \sigma_i^x - \gamma \sigma_i^z - \kappa) (\sigma_{i-1}^z + \sigma_{i+1}^z + 2\lambda), \quad (1)$$

where $i = 1, \dots, N$ are the sites of the lattice, $\sigma_i^{x,y,z}$ are the usual Pauli operators acting on site i , and the parameters $(\Gamma, \gamma, \kappa, \lambda)$, which quantify the strength of the various fields and couplings, are uniform throughout the lattice (i.e., there is *no quenched disorder*). We write the Hamiltonian as in (1) to highlight the connection to so-called kinetically constrained models of glasses [13]. When $\kappa + \gamma = \Gamma^2$ and $\lambda = 1$ the operator H is the symmetrized version of the master operator of a Fredrickson-Andersen (FA) facilitated spin model [13–15]. In this case, the first factor in each term of (1) is the local operator that flips a spin at site i . The second factor constrains the rate at site i to the state of the spins at sites $i \pm 1$, so that changes at site i cannot occur if both these spins are in the down state (in the z basis). For $\lambda > 1$ this constraint is softened, and we have the so-called soft-FA model, see [16]. When $\gamma + \kappa \neq \Gamma^2$, the operator H is related to a deformation of the master operator for the dynamics of the FA (or soft-FA) model, which is known to show a singular change in its ground state at some value of Γ [9, 16], as we discuss below.

We parametrize the couplings in the following way,

$$\Gamma = e^{-s} \sqrt{\epsilon}, \quad \gamma = \frac{1}{2}(1 - \epsilon), \quad \kappa = \frac{1}{2}(1 + \epsilon). \quad (2)$$

When $s = 0$, H is equivalent to a classical stochastic operator. In the quantum language we would say that in this case we are at a Rokhsar-Kivelson point [17, 18]: the ground-state, whose energy vanishes, is given by the “square root” of the equilibrium probability of the stochastic process, in this case the direct product,

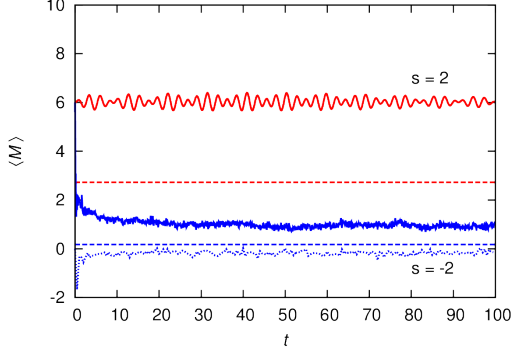


FIG. 1. *Relaxation of an observable.* Time evolution of the average magnetisation, $\langle M \rangle_t$. The system has size $N = 16$, with $\epsilon = 0.7$ and $\lambda = 1$. We show $\langle M \rangle_t$ from an initial state with well defined magnetisation M_0 at different values of s either side of $s_* = 0$: $s = -2$ with $M_0 = 6$ (full blue curve), $s = 2$ with $M_0 = 6$ (full red curve), and $s = -2$ with $M_0 = 0$ (dotted blue curve). The dashed lines indicate the average magnetisation expected if ETH holds (see Fig. 2). For the two curves at $s < 0$, $\langle M \rangle_t$ tends to values close to the ETH prediction at long times, while for $s > 0$ it does not.

$|\sqrt{\epsilon}\mathbf{e}_i\rangle \equiv (1+\epsilon)^{-N/2} \bigotimes_i (|0_i\rangle + \sqrt{\epsilon}|1_i\rangle)$, where $|n_i\rangle = |0, 1\rangle$ are the eigenstates of σ_i^z with eigenvalues -1 and $+1$, respectively, with ϵ the relative weight between a spin being up to being down in this basis.

We now expand on the connection to a stochastic problem when $s = 0$. The master operator that generates the stochastic dynamics of the FA model is [15, 16]

$$W = \sum_i [\epsilon\sigma_i^+ + \sigma_i^- - \epsilon\bar{n}_i - n_i] (n_{i-1} + n_{i+1} + \lambda - 1),$$

where $\sigma_i^z = 2n_i - 1$ and $\bar{n}_i = 1 - n_i$. In the classical problem the rate ϵ is typically determined by temperature, e.g., $\epsilon = e^{-1/T}$. The equilibrium state of W is $|\text{eq.}\rangle \propto \bigotimes_i (|0_i\rangle + \epsilon|1_i\rangle)$. From this one can construct the diagonal operator $P = \bigotimes_i (|0_i\rangle\langle 0_i| + \epsilon^{-1/2}|1_i\rangle\langle 1_i|)$, so that the Hermitian H is obtained from the stochastic W via a similarity transformation, $H = -P^{-1}WP$ (and thus have the same spectrum). This same transformation can be applied even when W is deformed away from being strictly stochastic by multiplying the terms with σ^\pm by a factor (thus breaking detailed balance) [9], giving the general Hamiltonian (1) for arbitrary s .

When $s \neq 0$ we are away from the RK point and the ground-state energy of H may no longer vanish. As s is increased, and depending on the value of ϵ , there can be a change in the ground-state which in the large size limit, $N \rightarrow \infty$, becomes singular at some $s_*(\lambda)$. For $\lambda \rightarrow 1$ this occurs for all ϵ at $s_* \rightarrow 0$, where the change is from the “equilibrium” state $|\sqrt{\epsilon}\mathbf{e}_i\rangle$, which dominates at $s = 0^-$, to the “inactive state” $|\text{in.}\rangle \approx \bigotimes_i |0_i\rangle$, which dominates at $s = 0^+$ [19]. This quantum phase transition in H is first-order [9, 16].

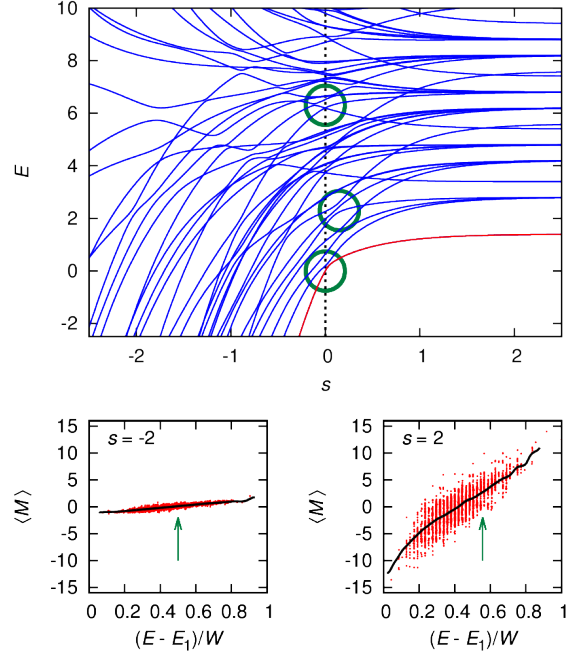


FIG. 2. *Spectral transitions and ETH.* (a) Energy spectrum of H for a system of size $N = 9$, with $\epsilon = 0.7$ and for $\lambda = 1$. We only show eigenvalues with zero momentum, and have removed the disconnected state $\bigotimes_i |0_i\rangle$ (see text). Close $s_* = 0$ the ground state energy (red) changes rapidly: the avoided crossing becomes a phase transition in the limit of large system size. Similar crossovers seem to occur near $s_* = 0$ in states throughout the spectrum, as indicated by the avoided crossings circled. (b,c) Expectation of the magnetization for each eigenstate $|E\rangle$, i.e., $M(E) = \langle E | \sum_i \sigma_i^z | E \rangle$, as a function of energy E (shifted to the centre of the spectrum, E_1 , and normalised by the bandwidth, W , for clarity), for s at either side of the putative MBL transition at $s = 0$ (system size $N = 16$ here). The black curve gives the corresponding microcanonical average, see text.

The ground-state transition from $|\sqrt{\epsilon}\mathbf{e}_i\rangle$ to $|\text{in.}\rangle$ at $s_* = 0$ (we consider $\lambda = 1$ from now on for simplicity) has the flavour of a localisation transition in the Fock basis $|\mathbf{n}\rangle \equiv \bigotimes_i |n_i\rangle$, since $|\sqrt{\epsilon}\mathbf{e}_i\rangle$ is spread as a probability over the states $|\mathbf{n}\rangle$ while $|\text{in.}\rangle$ is highly concentrated on $|0 \cdots 0\rangle$. Furthermore, in the context of the stochastic dynamics of constrained models this is a non-equilibrium transition from a relaxing “liquid” to a non-ergodic “glass” [9, 12]. So a natural question to ask is whether this change in behaviour extends higher up in the spectrum beyond the ground state and how does such a transition manifest itself in the context of a closed quantum system.

We first consider the thermalisation properties of this system by examining the time evolution of global observables under the unitary dynamics generated by H . Figure 1 shows the average total magnetisation in the z -direction, $\langle M \rangle_t$, as a function of time, where $M \equiv \sum_i \sigma_i^z$, and $\langle \cdot \rangle_t$ indicates expectation value in the state $|\psi(t)\rangle =$

$e^{-itH}|\text{init.}\rangle$ (where $\hbar = 1$). In Fig. 1 the initial state $|\text{init.}\rangle$ for all curves is a zero momentum state with well defined total magnetisation M_0 (i.e. the state obtained by uniform translating a Fock state with this magnetisation). The figure shows that there are two clearly distinct regimes depending on the value s . For $s < 0$, the average magnetisation $\langle M \rangle_t$ relaxes on a relatively short time scale (much shorter than the renewal timescale for the finite N systems we consider numerically), as shown for two initial states, $M_0 = 6$ (full blue curve) and $M_0 = 0$ (dotted blue curve). These two initial states were also chosen so that their average energies $\langle \text{init.} | H | \text{init.} \rangle$ energies were as close as possible in this finite sized system. The value to which their average magnetisation tends in the long time is also close, despite the fact that M_0 is very different. One would associate this behaviour with conditions where these observables thermalise [7] to a level dictated by their average energy. In contrast, for $s > 0$, the magnetisation remains close to the value in the initial state for long times, a behaviour one would associate with absence of thermalisation. This change in dynamical behaviour takes place at $s_* = 0$, the value of s of the quantum phase transition in the ground state of H . Note that the $|\text{init.}\rangle$ chosen are “atypical” initial states [6], in the sense that they are very different from the kind of states that are typically encountered in a thermal mixture. Similar change in the dynamics of $\langle M \rangle_t$ when crossing s_* is observed for other atypical initial conditions.

The result of Fig. 1 is a first indication that the ground-state transition at $s_* = 0$ may actually affect the bulk of the spectrum as well. To illustrate this we show in Fig. 2(a) the spectrum of H (for a smaller system of size $N = 9$, for clarity), as a function of s for fixed ϵ and $\lambda = 1$. The ground state is indicated by red. We show only the zero momentum sector to avoid trivial eigenvalue crossings due to the fact that the system is translationally invariant. We have also removed the isolated state $\bigotimes_i |0_i\rangle$ [19]. [Some crossings remain in Fig. 2(a) due to residual discrete symmetries which are more difficult to remove from the spectrum.] Around $s = 0$ there is a crossover, associated with the avoided crossing indicated by a circle. When $N \rightarrow \infty$ this crossover becomes the quantum phase transition of H at $s_* = 0$, related to the singular change of ground state from $|\sqrt{eq.}\rangle$ to $|\text{in.}\rangle$ [9, 16]. From Fig. 2(a) we also see that near $s = 0$ there seems to be a proliferation of other avoided crossings (also indicated by circles) in other eigenstates. With increasing N these features become sharper, which suggests that in the large N limit the first-order singularity of the ground-state at s_* will also be present throughout the spectrum.

If the change occurring in the spectrum at $s_* = 0$ is related to a MBL transition, one would expect also a change in the nature of the eigenstates. In particular, if $s < 0$ corresponds to a thermalising phase, eigenstates

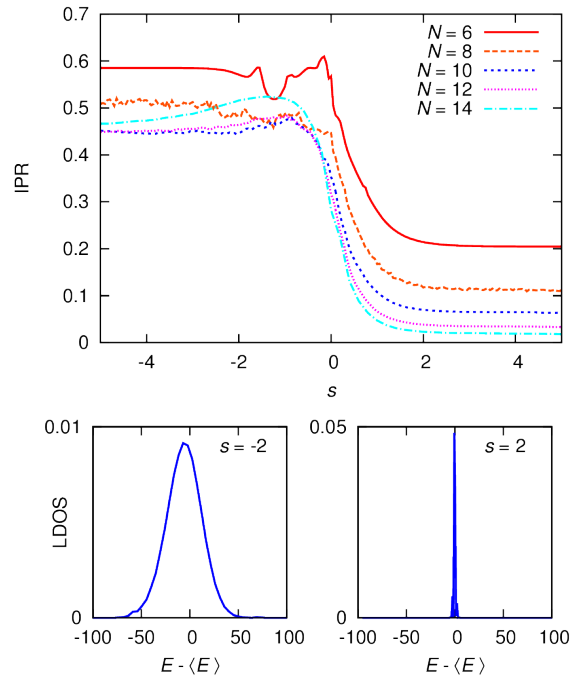


FIG. 3. *Localisation properties of eigenstates.* (a) Average inverse participation ratio (see text for definition) as a function of s for various system sizes. The average IPR changes from a value of $O(1)$, indicative of delocalised eigenstates in the Fock basis, for $s < 0$, to a value $O(1/N)$, indicative of localised eigenstates. This change becomes sharper with increasing system size, which again suggests a MBL transition in the large size limit at $s_* = 0$. (b) Average local density of states at two values of s at either side of the MBL transition: for $s < 0$ the LDOS is approximately Gaussian, as expected for a thermalising phase, while for $s > 0$ is highly peaked, as expected for a MBL phase.

should obey [6, 20] the eigenstate thermalisation hypothesis (ETH) [21, 22], at least in the bulk of the spectrum. Figures 2(b,c) test this in the case where the observable of interest is the magnetisation M . The symbols show the expectation value of the magnetisation for each eigenstate as a function of energy for system size $N = 16$. For $s < 0$, Fig. 2(b), this scatter plot is relatively smooth, and the average of M in eigenstates is close to the microcanonical average, as one would expect if ETH is obeyed (the black curve gives the microcanonical expectation value of M at each energy E [23]). In contrast, for $s > 0$, Fig. 2(c), there are very large variations in the eigenstate expectation values of M with energy, thus the eigenstate and microcanonical averages of M are not necessarily close to each other, indicative of a breakdown of ETH. We can contrast Figs. 2(b,c) with Fig. 1. The arrows in Figs. 2(b,c) indicate the energies of the initial states of Fig. 1, and the dashed lines in Fig. 1 give the values of the average magnetisation predicted if ETH holds. We now see that the level to which $\langle M \rangle_t$ tends at long times is close to that expected from ETH for $s < 0$, while

for $s > 0$ it is not.

Examining the spectrum in Fig. 2(a), we find in the non-thermalising phase ($s > 0$) the energy eigenstates form clusters, where every eigenstate within each cluster has a well defined magnetisation $\langle M \rangle$ similar to a subset of Fock states. This is an indication that in the non-thermalising phase the energy eigenstates may possess characteristics of the many-body Fock basis and that the long time dynamics (dominated by the intracluster energy eigenvalues) only have contributions from a small subset of Fock states with similar $\langle M \rangle$.

Our final test of a MBL transition in the spectrum of H is by considering how the eigenstates are distributed over the Fock basis states $|\mathbf{n}\rangle$. We quantify this as usual by means of the inverse participation ratio (IPR), which for an eigenstate $|E\rangle$ reads, $I(E) \equiv 2^{-N} / \sum_{\mathbf{n}} |\langle E|\mathbf{n}\rangle|^4$ [such that $I(E)$ is $O(2^{-N})$ if $|E\rangle$ is highly concentrated on some $|\mathbf{n}\rangle$, and $O(1)$ otherwise]. Figure 3(a) shows the average of the IPR over the bulk of the spectrum, $\bar{I} \equiv 2^{-N} \sum_E I(E)$, as a function of s [24]. There is a clear crossover in the behaviour of \bar{I} occurring near $s_* = 0$. For $s < 0$ the average IPR is of $O(1)$, indicative of eigenstates spread out in $|\mathbf{n}\rangle$. For $s > 0$ in contrast, \bar{I} becomes small, seemingly tending to zero progressively with N , indicative of eigenstates localised in $|\mathbf{n}\rangle$. The change also sharpens with increasing N , which is consistent with a discontinuity at $s_* = 0$ in the large size limit. In Figs. 3(b,c) we show the average local density of states (LDOS). This is defined from the local density of states, $\text{LDOS}(E|\mathbf{n}) \equiv \sum_{E'} |\langle E'|\mathbf{n}\rangle|^2 \delta(E - E')$, by centring each $\text{LDOS}(E|\mathbf{n})$ around its average E and then averaging over \mathbf{n} . The average LDOS is shown for s at either side of s_* : for $s < 0$ the distribution over eigenstates appears Gaussian, while for $s > 0$ it is highly concentrated on an eigenstate. This indicates that in the non-thermalising phase the eigenstates of the system are localised over a few many-body Fock states with that energy. The crossover from a distribution which is close to a delta function to one which is Gaussian is similar to the change one would encounter in the LDOS of an integrable quantum system (where thermalisation is not expected to occur) when a banded non-integrable perturbation is added (such that the perturbed system would thermalise) [25]. In such a system one would expect that the average IPR \bar{I} is approximately $\frac{\sqrt{2}}{3}$ (~ 0.47) (note that this is less than the \bar{I} which one would expect for random eigenstates described by the Porter-Thomas distribution). The extracted \bar{I} is close to this value, highlighting the connection between the LDOS, the average IPR and non-integrable perturbations in integrable systems.

The evidence above suggests that in the quantum system described by the Hamiltonian (1), in the case where $\lambda = 1$, there is a MBL transition occurring at $s_* = 0$ for all values of the parameter ϵ . As emphasised above, H is also (minus) the Hermitian version of the deformed

(or tilted [26]) generator for the dynamics of the FA glass model [9]. The classical FA model is known to have a first-order dynamical (or “space-time”) transition at $s_0 = 0$ from a state of relaxing and active dynamics to one of non-relaxing inactive dynamics, i.e. a non-equilibrium glass state [9]. Our results here indicate that this non-equilibrium glass transition is present throughout the spectrum, and is the same transition giving rise to MBL in the quantum problem.

For the case where $\lambda > 1$, the Hamiltonian (1) is associated to the soft-FA model [16]. From the classical soft-FA dynamics we know that in this case H may or may not have a quantum phase transition in its ground state depending on λ and ϵ . For $\lambda - 1 \gtrsim 0$ the ground state transition is present [16], and we therefore expect a MBL transition across the spectrum similar to the $\lambda = 1$ case, but occurring at some $s_*(\lambda, \epsilon) \neq 0$. In contrast, when $\lambda - 1$ is large enough the ground state transition disappears [16], which allows for an interesting situation: in the quantum problem we may expect to see that in this case the MBL transition only occurs for states beyond an inverted “mobility edge” [6].

Other kinetically constrained glass models have active-inactive transitions [9, 16]. One such class of systems which would be interesting to consider in the quantum context are constrained lattice gases [13], where hopping between sites is constrained by the state of neighbouring sites. If their associated quantum problem also displays a MBL transition, like the one we showed here for systems based on facilitated spin models of glasses, then this MBL transition would be one where also particle transport ceases in the localised phase.

This work was supported by Leverhulme Trust grant No. F/00114/BG.

-
- [1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
 - [2] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
 - [3] D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. **321**, 1126 (2006); D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Phys. Rev. B **76**, 052203 (2007).
 - [4] V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007); A. Pal and D. A. Huse, *ibid.* **82**, 174411 (2010).
 - [5] M. Žnidarič, T. Prosen, and P. Prelovšek, Phys. Rev. B **77**, 064426 (2008); C. Monthus and T. Garel, *ibid.* **81**, 134202 (2010); R. Vosk and E. Altman, Phys. Rev. Lett. **110**, 067204 (2013); Y. B. Lev and D. R. Reichman, arXiv:1402.0502.
 - [6] For a recent review see R. Nandkishore and D. A. Huse, arXiv:1404.0686.
 - [7] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
 - [8] T. Grover and M. P. A. Fisher, arXiv:1307.2288; W. De Roeck and F. Huveneers, arXiv:1308.6263; M. Schiulaz and M. Müller, arXiv:1309.1082.

- [9] J. P. Garrahan, R. L. Jack, V. Lecomte, E. Pitard, K. van Duijvendijk, and F. van Wijland, Phys. Rev. Lett. **98**, 195702 (2007); J. Phys. A **42**, 075007 (2009).
- [10] V. Lecomte, C. Appert-Rolland, and F. van Wijland, J. Stat. Phys. **127**, 51 (2007).
- [11] M. Baiesi, C. Maes, and B. Wynants, Phys. Rev. Lett. **103**, 010602 (2009).
- [12] L. O. Hedges, R. L. Jack, J. P. Garrahan, and D. Chandler, Science **323**, 1309 (2009).
- [13] For reviews see: F. Ritort and P. Sollich, Adv. Phys. **52**, 219 (2003); J. P. Garrahan, P. Sollich, and C. Toninelli, arXiv:1009.6113.
- [14] G. H. Fredrickson and H. C. Andersen, Phys. Rev. Lett. **53**, 1244 (1984).
- [15] R. L. Jack, P. Mayer, and P. Sollich, J. Stat. Mech. , P03006 (2006).
- [16] Y. S. Elmatad, R. L. Jack, D. Chandler, and J. P. Garrahan, Proc. Natl. Acad. Sci. USA **107**, 12793 (2010).
- [17] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
- [18] C. Castelnovo, C. Chamon, C. Mudry, and P. Pujol, Ann. Phys. **318**, 316 (2005).
- [19] Note that when $\lambda = 1$ the state $\bigotimes_i |0_i\rangle$ is an eigenstate with zero eigenvalue for all s . This single state forms an isolated subspace of H of dimension one, and we remove it from the spectrum. The quantum phase transition occurs in the “connected component” of H : the support of $|\text{in.}\rangle$ is on states $|\mathbf{n}\rangle$ with a subextensive number of $n_i = 1$, and the difference between $|\text{in.}\rangle$ and $\bigotimes_i |0_i\rangle$ becomes vanishingly small in the large size limit [9, 16]. For $\lambda > 1$, $\bigotimes_i |0_i\rangle$ ceases to be a trivial eigenstate, and all $|\mathbf{n}\rangle$ are connected by H .
- [20] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Phys. Rev. Lett. **96**, 050403 (2006).
- [21] J. Deutsch, Phys. Rev. A **43**, 2046 (1991); M. Srednicki, Phys. Rev. E **50**, 888 (1994); H. Tasaki, Phys. Rev. Lett. **80**, 1373 (1998).
- [22] See also: M. Rigol, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008); M. Rigol and M. Srednicki, Phys. Rev. Lett. **108**, 110601 (2012); W. Beugeling, R. Moessner, and M. Haque, Phys. Rev. E **89**, 042112 (2014); R. Steinigeweg, A. Khodja, H. Niemeyer, C. Gogolin, and J. Gemmer, **112**, 130403 (2014); S. Khlebnikov and M. Kruczenski, arXiv:1312.4612.
- [23] The microcanonical density matrix at an energy E is estimated as a mixture of eigenstates with occupation probabilities given by a Gaussian centred on E of width $\sim 200\Delta$, where Δ is the mean energy level spacing.
- [24] The average is over half of all eigenstates around the centre of the spectrum.
- [25] S. Genway, A. F. Ho, and D. K. K. Lee, Phys. Rev. Lett. **111**, 130408 (2013); V. V. Flambaum and F. M. Izrailev, Phys. Rev. E **61**, 2539 (2000); Phys. Rev. E **64**, 026124 (2001).
- [26] H. Touchette, Phys. Rep. **478**, 1 (2009).